XXV. *A new Method of inveſtigating the Sums of infinite Series. By the Rev.* S. VINCE, *A. M. of* CAMBRIDGE, *in a Letter to* Henry Maty, *A. M. Secretary.*

Read June 6, 1782.

SIR,

HAVING lately diſcovered ſome very eaſy methods of inveſtigating the ſums of certain infinite ſeries, I have taken the liberty of requeſting the favour of you to preſent them to the Royal Society. I have divided the ſubject into three parts: the firſt contains a new and general method of finding the ſum of thoſe ſeries which DE MOIVRE has found in one or two particular caſes; but whoſe method, although it be in appearance general, will, upon trial, be found to be abſolutely impracticable. The ſecond contains the ſummation of certain ſeries, the laſt differences of whoſe numerators become equal to nothing. The third contains obſervations on a *correction* which is neceſſary in inveſtigating the ſums of certain ſeries by collecting two terms into one, with its application to a variety of caſes.

I am, &c.

Cambridge,
May 3, 1782.

PART

PART I.

LEM. I.

Let r be any whole number, and then the fluent of $\frac{\dot{x}}{1+x^r}$ *can always be exhibited by circular arcs and logarithms; but when* $x=1$, *the fluent of the same fluxion will be expressed by the infinite series* $1-\frac{1}{r+1}+\frac{1}{2r+1}-\frac{1}{3r+1}+$&c. *the sum of this series therefore can always be found by circular arcs and logarithms.*

LEM. II.

To find the sum of the infinite series $\frac{a}{1.r+1}-\frac{a+b}{r+1.2r+1}+\frac{a+2b}{2r+1.3r+1}-$&c.

Assume $1-\frac{1}{r+1}+\frac{1}{2r+1}-\frac{1}{3r+1}+$ &c. . $=S$; therefore,

(A) $1-\frac{1}{1.r+1}+\frac{r+1}{r+1.2r+1}-\frac{2r+1}{2r+1.3r+1}+$&c. . . . $=S$.

In the first series, add together the 1st and 2d, the 2d and 3d, &c. &c. terms, and the resulting series will evidently be equal to twice that series *minus* the first term; therefore,

(B), $\frac{r}{1.r+1}-\frac{r}{r+1.2r+1}+\frac{r}{2r+1.3r+1}-$&c. . . . $=2S-1$.

Now $\left(\frac{A}{r}\right)\frac{1}{r}-\frac{\frac{1}{r}}{1.r+1}+\frac{1+\frac{1}{r}}{r+1.2r+1}-\frac{2+\frac{1}{r}}{2r+1.3r+1}+$&c. . . . $=\frac{S}{r}$,

or $\left.\begin{array}{l}\frac{1}{r+1.2r+1}-\frac{2}{2r+1.3r+1}+\text{\&c.} \ldots \\ +\frac{1}{r}-\frac{\frac{1}{r}}{1.r+1}+\frac{\frac{1}{r}}{r+1.2r+1}-\text{\&c.} \ldots\end{array}\right\}=\frac{S}{r}$;

Now

Now the ſum of the lower ſeries, omitting the firſt term, is equal to $-$B divided by r^2, or $= -\frac{2S-1}{r^2}$; hence, by tranſpoſition, and, multiplying both ſides by b, we ſhall have,

$$\frac{b}{\overline{r+1}\,.\,\overline{2r+1}} - \frac{2b}{\overline{2r+1}\,.\,\overline{3r+1}} + \&c.\ldots = \frac{bS}{r} + \frac{2bS - \overline{r+1}\,.\,b}{r^2};$$ alſo by multiplying B by $\frac{a}{r}$ we have

$$\frac{a}{1\,.\,\overline{r+1}} - \frac{a}{\overline{r+1}\,.\,\overline{2r+1}} + \frac{a}{\overline{2r+1}\,.\,\overline{3r+1}} - \&c.\ldots = \frac{2aS-a}{r};$$ ſubtract the laſt equation but one from the laſt, and we ſhall have

$$\frac{a}{1\,.\,\overline{r+1}} - \frac{a+b}{\overline{r+1}\,.\,\overline{2r+1}} + \frac{a+2b}{\overline{2r+1}\,.\,\overline{3r+1}} - \&c.\ldots = \frac{\overline{2ra - \overline{r+2}\,.\,b} \times S - ra + \overline{r+1}\,.\,b}{r^2}$$

Cor. 1. Hence it appears, that the ſum of this ſeries can never be exhibited in finite terms, except $a : b$ as $r+2 : 2r$, in which caſe the ſum is equal to $\frac{a}{r+2}$.

Hence, if $a=3$, $b=2$, then $r=1$; $\therefore \frac{3}{1.2} - \frac{5}{2.3} + \frac{7}{3.4} - \&c.\ldots = 1$;

if $a=1$, $b=4$, then $r=\frac{4}{3}$; $\therefore \frac{5}{3.7} - \frac{9}{7.11} + \frac{13}{11.15} - \frac{17}{15.19} + \&c.\ldots = \frac{1}{6}$;

if $a=4$, $b=3$, then $r=\frac{6}{5}$; $\therefore \frac{4}{5.11} - \frac{7}{11.17} + \frac{10}{17.23} - \frac{13}{23.29} + \&c.\ldots = \frac{1}{20}$;

Cor 2. Put $a=c-b$, and we ſhall have, after tranſpoſition,

$$\frac{c}{\overline{r+1}\,.\,\overline{2r+1}} \quad \frac{c+b}{\overline{2r+1}\,.\,\overline{3r+1}} + \&c.\ldots = \frac{\overline{3r+2\,.\,b - 2rc} \times S - \overline{2r+1}\ b + rc}{r^2} + \frac{c-b}{r+1}.$$

P R O P. I.

To find the ſum of the infinite ſeries $\frac{m}{1\,.\,\overline{r+1}\,.\,\overline{2r+1}} + \frac{m+n}{\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}} - \frac{m+2n}{\overline{4r+1}\,.\,\overline{5r+1}\,.\,\overline{6r+1}} + \&c.$

Every

Every series of this kind may be resolved into the following series $\frac{a}{1\cdot\overline{r+1}} - \frac{a+b}{\overline{r+1}\cdot\overline{2r+1}} + \frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}} - \frac{a+3b}{\overline{3r+1}\cdot\overline{4r+1}} + \&c.$ for if we reduce two terms of this series into one, it will become

$$\frac{2ar-b}{1\cdot\overline{r+1}\cdot\overline{2r+1}} + \frac{2ra+\overline{2r-1}\cdot b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}} + \frac{2ra+\overline{4r-1}\cdot b}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}} + \&c.$$

where the denominators being the same as in the given series, and the numerators also in arithmetic progression, we have only to take a and b such quantities that the respective numerators may be also equal; assume, therefore, $2ra-b=m$, $2ra+\overline{2r-1}\cdot b=m+n$; therefore, $b=\frac{n}{2r}$, $a=\frac{2rm+n}{4r^2}$, which substituted for a and b in LEM. 2. gives

$$\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}} + \frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}} + \frac{m+2n}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}} + \&c.\ldots$$
$$=\frac{2rm-\overline{r+1}\cdot n}{2r^3}\times S+\frac{\overline{2r+1}\cdot\overline{n-2rm}}{4r^3}.$$

Let $r=1$, and we have

$$\frac{m}{1\cdot2\cdot3}+\frac{m+n}{3\cdot4\cdot5}+\frac{m+2n}{5\cdot6\cdot7}+\&c.\ldots=\overline{m-n}\cdot S+\frac{3n-2m}{4}.$$

If $m=1$, $n=3$, $\frac{1}{1\cdot2\cdot3}+\frac{4}{3\cdot4\cdot5}+\frac{7}{5\cdot6\cdot7}+\&c.\ldots=\frac{7}{4}-2S$;

$m=1$, $n=0$, $\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1}{5\cdot6\cdot7}+\&c.\ldots=S-\frac{1}{2}$.

Let $r=2$, and the series becomes

$$\frac{m}{1\cdot3\cdot5}+\frac{m+n}{5\cdot7\cdot9}+\frac{m+2n}{9\cdot11\cdot13}+\&c.\ldots=\frac{4m-3n}{16}\times S+\frac{5n-4m}{32}.$$

If $m=1$, $n=1$, $\frac{1}{1\cdot3\cdot5}+\frac{2}{5\cdot7\cdot9}+\frac{3}{9\cdot11\cdot13}+\&c.\ldots=\frac{S}{16}+\frac{1}{32}$;

$m=1$, $n=0$, $\frac{1}{1\cdot3\cdot5}+\frac{1}{5\cdot7\cdot9}+\frac{1}{9\cdot11\cdot13}+\&c.\ldots=\frac{S}{4}-\frac{1}{8}$.

Let $r=5$, and we shall have

$$\frac{m}{1\cdot6\cdot11}+\frac{m+n}{11\cdot16\cdot21}+\frac{m+2n}{21\cdot26\cdot31}+\&c.\ldots=\frac{5m-3n}{125}\times S+\frac{11n-10m}{500}.$$

If $m=1$, $n=1$, $\frac{1}{1.6.11}+\frac{2}{11.16.21}+\frac{3}{21.26.31}+\&c.\ldots=\frac{2}{125}\times S+\frac{1}{500}$;

$m=1$, $n=0$, $\frac{1}{1.6.11}+\frac{1}{11.16.21}+\frac{1}{21.26.31}+\&c.\ldots=\frac{S}{25}-\frac{1}{50}$.

Cor. If $2r : r+1 :: n : m$, the sum of the series can be accurately found, and will be equal to $\frac{m}{2r\cdot\overline{r+1}}$. Let therefore $m=r+1$, and then $n=2r$, consequently

$$\frac{1}{1\cdot\overline{2r+1}}+\frac{1}{\overline{2r+1}\cdot\overline{4r+1}}+\frac{1}{\overline{4r+1}\cdot\overline{6r+1}}+\&c.\ldots=\frac{1}{2r};$$

which is also known from other principles.

P R O P. II.

To find the sum of the infinite series $\frac{m}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{m+n}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\frac{m+2n}{\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}}+\&c.$

This series resolves itself into

$$\frac{c}{\overline{r+1}\cdot\overline{2r+1}}-\frac{c+b}{\overline{2r+1}\cdot\overline{3r+1}}+\frac{c+2b}{\overline{3r+1}\cdot\overline{4r+1}}-\&c.;$$

for by reduction, as before, it becomes

$$\frac{2cr-\overline{r+1}\cdot b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{2cr+\overline{r-1}\cdot b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\frac{2cr+\overline{3r-1}\cdot b}{\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}}+\&c.$$

where the denominators are the same as in the given series, and the numerators in arithmetic progression; assume therefore $2cr-\overline{r+1}\cdot b=m$, $2cr+\overline{r-1}\cdot b=m+n$, hence $b=\frac{n}{2r}$, $c=\frac{2rm+\overline{r+1}\cdot n}{4r^2}$, which, substituted in cor. 2. LEM. 2. give

$$\frac{m}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}+\frac{m+n}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}+\frac{m+3n}{\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}}+\&c.\ldots$$

$$=\frac{\overline{2r+1}\cdot n-2rm}{2r^3}\times S+\frac{2rm-\overline{3r+1}\cdot n}{4r^3}+\frac{2rm-\overline{r-1}\cdot n}{4r^2\cdot\overline{r+1}}.$$

 Cor.

Cor. 1. In prop. 1. substitute a for m, and $2b$ for n, and we have

$$\frac{a}{1.\overline{r+1}.\overline{2r+1}} + \frac{a+2b}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}} + \frac{a+4b}{\overline{4r+1}.\overline{5r+1}.\overline{6r+1}} + \&c.\ldots =$$
$$\frac{ra-\overline{r+1}.b}{r^3} \times S + \frac{\overline{2r+1}.b-ra}{2r^3}.$$

Also in this prop. substitute $a+b$ for m, and $2b$ for n, and we have

$$\frac{a+b}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}} + \frac{a+3b}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}} + \&c.\ldots =$$
$$\frac{\overline{r+1}.b-ra}{r^3} \times S + \frac{ra-\overline{2r+1}.b}{2r^3} + \frac{ra+b}{2r^2\times\overline{r+1}}.$$

Subtract this latter series from the former, and

$$\frac{a}{1.\overline{r+1}.\overline{2r+1}} - \frac{a+b}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}} + \frac{a+2b}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}} - \&c.\ldots =$$
$$\frac{2ra-\overline{r+1}.2b}{r^3} \times S + \frac{\overline{2r+1}.b-ra}{r^3} - \frac{ra+2b}{2r^2\times\overline{r+1}}.$$

Let $r=1$, and we have

$$\frac{a}{1.2.3} - \frac{a+b}{2.3.4} + \frac{a+2b}{3.4.5} - \&c.\ldots = \overline{2a-4b}\times S + \frac{11b-5a}{4}.$$

If $a=1$, $b=0$, $\frac{1}{1.2.3} - \frac{1}{2.3.4} + \frac{1}{3.4.5} - \&c.\ldots = 2S - \frac{5}{4}$;

$a=1$, $b=2$, $\frac{1}{1.2.3} - \frac{3}{2.3.4} + \frac{5}{3.4.5} - \&c.\ldots = \frac{17}{4} - 6S$.

Let $r=3$, and we have

$$\frac{a}{1.4.7} - \frac{a+b}{4.7.10} + \frac{a+2b}{7.10.13} - \&c.\ldots = \frac{6a-8b}{27}\times S + \frac{53b-33a}{216}.$$

If $a=1$, $b=0$, $\frac{1}{1.4.7} - \frac{1}{4.7.10} + \frac{1}{7.10.13} - \&c.\ldots = \frac{2}{9}S - \frac{11}{72}$;

$a=1$, $b=1$, $\frac{1}{1.4.7} - \frac{2}{4.7.10} + \frac{3}{7.10.13} - \&c.\ldots = \frac{5}{54} - \frac{2}{27}S$.

If, instead of substituting in prop. 1. $2b$ and a for n and m, we had substituted two other quantities, as $2r$ and s, and then proceeded as above, a series would have been formed, the numerators of whose alternate terms would have formed each a a separate arithmetic progression.

If the latter series had been *added* to the former a series would have been formed whose terms would have been all positive; but as I purpose, in the second part of this paper, to give a general method of summing all such series, I shall not stop here to apply this method of investigation.

Cor. 2. In proposition 2. substitute a for m, and $2b$ for n, and we shall have

$$\frac{a}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}} + \frac{a+2b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}} + \&c.\ldots = \frac{\overline{2r+1}\cdot b - ra}{r^3}\times S + \frac{ra-\overline{3r+1}\cdot b}{2r^3} + \frac{ra-\overline{r-1}\cdot b}{2r^2\cdot\overline{r+1}}.$$

Also in prop. 1. write $a-b$ for m, and $2b$ for n, and there results

$$\frac{a+b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}} + \frac{a+3b}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}} + \&c.\ldots = \frac{ra-\overline{2r+1}\cdot b}{r^3}\times S + \frac{\overline{r+1}\cdot b - ra}{2r^3} - \frac{a-b}{\overline{r+1}\cdot\overline{2r+1}}.$$

Subtract this latter series from the former, and we shall have

$$\frac{a}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}} - \frac{a+b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}} + \frac{a+2b}{\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}} - \&c.\ldots$$
$$= \frac{\overline{2r+1}\cdot 2b - 2ra}{r^3}\times S + \frac{ra-\overline{3r+1}\cdot b}{r^3} + \frac{ra-\overline{r-1}\cdot b}{2r^2\cdot\overline{r+1}} + \frac{a-b}{\overline{r+1}\cdot\overline{2r+1}}.$$

P R O P. III.

To find the sum of the infinite series $\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}} + \frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}} + \frac{m+2n}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}} + \&c.$

This series resolves itself into

$$\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}} - \frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}} + \frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}} - \&c.$$

for by reduction it becomes

$$\frac{3ra-b}{1.\overline{r+1}.\overline{2r+1}.\overline{3r+1}}+\frac{3ar+\overline{4r-1}.b}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}+$$
$$\frac{3ar+\overline{8r-1}.b}{\overline{4r+1}.\overline{5r+1}.\overline{6r+1}.\overline{7r+1}}+\&c.$$

where the numerators are in arithmetic progreſſion, and the denominators the ſame as in the given ſeries; aſſume therefore $3ra-b=m$, $3ra+\overline{4r-1}.b=m+n$, hence $b=\frac{n}{4r}$, $a=\frac{4rm+n}{12r^2}$; ſubſtitute theſe values into cor. 1. prop. 2. and we have

$$\frac{m}{1.\overline{r+1}.\overline{2\ +1}.\overline{3r+1}}+\frac{m+n}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}+$$
$$\frac{m+2n}{\overline{4r+1}.\overline{5r+1}.\overline{6r+1}.\overline{7r+1}}+\&c.\ldots=\frac{4rm-\overline{3r+2}.n}{6r^4}\times S+$$
$$\frac{\overline{3r+1}.n-2rm}{6r^4}-\frac{rm+n}{6r^3.\overline{r+1}}.$$

Let $r=1$, and we have

$$\frac{m}{1.2.3.4}+\frac{m+n}{3.4.5.6}+\frac{m+2n}{5.6.7.8}+\&c.\ldots=\frac{4m-5n}{6}\times S+\frac{7n-5m}{12}.$$

If $m=1, n=1$, $\frac{1}{1.2.3.4}+\frac{1}{3.4.5.6}+\frac{1}{5.6.7.8}+\&c.\ldots=\frac{2}{3}S-\frac{5}{12}$;

$m=1, n=1$, $\frac{1}{1.2.3.4}+\frac{2}{3.4.5.6}+\frac{3}{5.6.7.8}+\&c.\ldots=\frac{1}{6}-\frac{1}{6}S$;

$m=7, n=5$, $\frac{7}{1.2.3.4}+\frac{12}{3.4.5.6}+\frac{17}{5.6.7.8}+\&c.\ldots=\frac{1}{2}S$;

Let $r=2$, and we have

$$\frac{m}{1.3.5.7}+\frac{m+n}{5.7.9.11}+\frac{m2n}{9.11.13.15}+\&c.\ldots=\frac{m-n}{12}\times S-\frac{19n-16m}{288}.$$

If $m=1, n=3$, $\frac{1}{1.3.5.7}+\frac{4}{5.7.9.11}+\frac{7}{9.11.13.15}+\&c.\ldots=\frac{41}{288}-\frac{1}{6}S$;

$m=1, n=0$, $\frac{1}{1.3.5.7}+\frac{1}{5.7.9.11}+\frac{1}{9.11.13.15}+\&c.\ldots=\frac{1}{12}S-\frac{1}{18}$.

$m=19, n=16$. $\frac{19}{1.3.5.7}+\frac{35}{5.7.9.11}+\frac{51}{9.11.13.15}+\&c.\ldots=\frac{1}{4}S$.

Cor. If $n:m$ as $4r:3r+2$, the ſum of the ſeries can be accurately had; let therefore $n=4r$ and $m=3r+2$, and we ſhall have

$3r+2$

$$\frac{3r+2}{1.\overline{r+1}.\overline{2r+1}.\overline{3r+1}}+\frac{7r+2}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}+\&c.\ldots=\frac{1}{2r.\overline{r+1}}.$$

If $r=1$, $\frac{5}{1.2.3.4}+\frac{9}{3.4.5.6}+\frac{13}{5.6.7.9}+\&c.\ldots=\frac{1}{4}$ *;

$r=2$, $\frac{1}{1.3.5.7}+\frac{2}{5.7.9.11}+\frac{3}{9.11.13.15}+\&c.\ldots=\frac{1}{96}$;

$r=6$, $\frac{5}{1.7.13.19}+\frac{11}{13.19.25.31}+\frac{17}{25.31.37.43}+\&c.\ldots=\frac{1}{336}$.

PROP. IV.

To find the sum of the infinite series $\frac{m}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}.\overline{4r+1}}+\frac{m+n}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}.\overline{6r+1}}+\frac{m+2n}{\overline{5r+1}.\overline{6r+1}.\overline{7r+1}.\overline{8r+1}}+\&c.$

This series resolves itself into

$$\frac{a}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}}-\frac{a+b}{\overline{2r+1}.\overline{3r+1}.\overline{4r+1}}+\frac{a+2b}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}}-\&c.$$

for by reduction it becomes $\frac{3ra-\overline{r+1}.b}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}.\overline{4r+1}}+\frac{3ra+\overline{3r-1}.b}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}.\overline{6r+1}}+\frac{3ra+\overline{7r-1}.b}{\overline{5r+1}.\overline{6r+1}.\overline{7r+1}.\overline{8r+1}}+\&c.$ where the denominators are the same as in the given series, and the numerators also in arithmetic progression; put therefore $3ra-\overline{r+1}.b=m$, $3ra+\overline{3r-1}.b=m+n$, hence $b=\frac{n}{4r}$, $a=\frac{4rm+\overline{r+1}.n}{12r^2}$, which, substituted in cor. 2. prop. 2. give

$$\frac{m}{\overline{r+1}.\overline{2r+1}.\overline{3r+1}.\overline{4r+1}}+\frac{m+n}{\overline{3r+1}.\overline{4r+1}.\overline{5r+1}.\overline{6r+1}}+\&c.\ldots=$$
$$\frac{\overline{5r+2}.n-4rm}{6r^4}\times S+\frac{2rm-\overline{4r+1}.n}{6r^4}+\frac{2rm-\overline{r-2}.n}{12.r^3.\overline{r+1}}+\frac{4rm-\overline{2r-1}.n}{12r^2.\overline{r+1}.\overline{2r+1}}.$$

* Vide DE MOIVRE's Mis. Anal. pag. 134.

Cor.

Cor. 1. In prop. 3. write a for m and $2b$ for n, and we have

$$\frac{a}{1\,.\,\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}}+\frac{a+2b}{\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}\,.\,\overline{5r+1}}+\&c.\ldots=$$
$$\frac{2ra-\overline{3r+2}\,.\,b}{3r^4}\times S+\frac{\overline{3r+1}\,.\,b-ra}{3r^4}-\frac{ra+2b}{6r^3\,.\,\overline{r+1}}.$$

Also in this prop. write $a+b$ for m, and $2b$ for n, and we have

$$\frac{a+b}{\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}}+\frac{a+3b}{\overline{3r+1}\,.\,\overline{4r+1}\,.\,\overline{5r+1}\,.\,\overline{6r+1}}+\&c.\ldots=$$
$$\frac{\overline{3r+2}\,.\,b-2ra}{3r^4}\times S+\frac{ra-\overline{3r+1}\,.\,b}{3r^4}+\frac{ra+2b}{6r^3\,.\,\overline{r+1}}+\frac{2ra+b}{6r^2\,.\,\overline{r+1}\,.\,\overline{2r+1}}.$$

subtract this latter series from the former, and we have

$$\frac{a}{1\,.\,\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}}-\frac{a+b}{\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}}+\&c.\ldots=$$
$$\frac{4ra-\overline{3r+2}\,.\,2b}{3r^4}\times S+\frac{\overline{3r+1}\,.\,2b-2ra}{3r^4}-\frac{ra+2b}{3r^3\,.\,\overline{r+1}}-\frac{2ra+b}{6r^2\,.\,\overline{r+1}\,.\,\overline{2r+1}}.$$

Let $r=1$, and we have

$$\frac{a}{1.2.3.4}-\frac{a+b}{2.3.4.5}+\frac{a+2b}{3.4.5.6}-\&c.\ldots=\frac{4a-10b}{3}\times S+\frac{83b-32a}{36}.$$

If $a=1$, $b=0$, $\frac{1}{1.2.3.4}-\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}-\&c.\ldots=\frac{4}{3}S-\frac{8}{9}.$

$a=3$, $b=1$, $\frac{1}{1.2.4}-\frac{1}{2.3.5}+\frac{1}{3.4.6}-\&c.\ldots=\frac{2}{3}S-\frac{13}{36}.$

Let $r=3$, and we have

$$\frac{a}{1.4.7.10}-\frac{a+b}{4.7.10.13}+\frac{a+2b}{7.10.13.16}-\&c.\ldots=\frac{12a-22b}{243}\times S+\frac{1027b-156a}{3.81.56}$$

If $a=1$, $b=1$, $\frac{1}{1.4.7.10}-\frac{2}{4.7.10.13}+\frac{3}{7.10.13.16}-\&c.\ldots=\frac{871}{3.81.56}-\frac{10}{3.81}S.$

$a=4$, $b=3$, $\frac{1}{1.7.10}-\frac{1}{4.10.13}+\frac{1}{7.13.16}-\&c.\ldots=\frac{13}{72}-\frac{2}{27}S.$

Cor. 2. If $a:b$ as $3r+2:2r$, the sum of the series can be accurately found; take $\therefore a=3r+2$, and $b=2r$, and we shall have

$$\frac{3r+2}{1\,.\,\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}}-\frac{5r+2}{\overline{r+1}\,.\,\overline{2r+1}\,.\,\overline{3r+1}\,.\,\overline{4r+1}}+\&c.\ldots=\frac{1}{\overline{r+1}\,.\,\overline{2r+1}}.$$

If

If $r=1$, $\frac{5}{1\cdot2\cdot3\cdot4}-\frac{7}{2\cdot3\cdot4\cdot5}+\frac{9}{3\cdot4\cdot5\cdot6}-\&c.\ldots=\frac{1}{60}$;

$r=2$, $\frac{2}{1\cdot3\cdot5\cdot7}-\frac{3}{3\cdot5\cdot7\cdot9}+\frac{4}{5\cdot7\cdot9\cdot11}-\&c.\ldots=\frac{1}{15}$;

$r=3$, $\frac{11}{1\cdot4\cdot7\cdot10}-\frac{17}{4\cdot7\cdot10\cdot13}+\frac{23}{7\cdot10\cdot13\cdot16}-\&c.\ldots=\frac{1}{28}$.

P R O P. V.

To find the ſum of the infinite ſeries $\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\frac{m+2n}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}\cdot\overline{8r+1}}+\&c.$

This ſeries reſolves itſelf into $\frac{a}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}}-\frac{a+b}{\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{a+2b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}}-\&c.$; for by reduction this ſeries becomes $\frac{4ra-b}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{4ra+\overline{6r-1}\cdot b}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\frac{4ra+\overline{12r-1}\cdot b}{\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}\cdot\overline{7r+1}\cdot\overline{8r+1}}+\&c.$ where the numerators are in arithmetic progreſſion, and the denominators the ſame as in the given ſeries; aſſume therefore $4ra-b=m$, $4ra+\overline{6r-1}\cdot b=m+n$, hence $b=\frac{n}{6r}$, $a=\frac{6rm+n}{24r^2}$, which values being ſubſtituted in cor. 1. prop. 4. give

$$\frac{m}{1\cdot\overline{r+1}\cdot\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}}+\frac{m+n}{\overline{2r+1}\cdot\overline{3r+1}\cdot\overline{4r+1}\cdot\overline{5r+1}\cdot\overline{6r+1}}+\&c.\ldots=$$

$$\frac{2rm-\overline{2r+1}\cdot n}{6r^5}\times S+\frac{\overline{4r+1}\cdot n-2rm}{12r^5}-\frac{6rm+9n}{72\cdot r^4\cdot\overline{r+1}}-\frac{2rm+n}{24r^3\cdot\overline{r+1}\cdot\overline{2r+1}}.$$

Let $r=1$, and we have

$$\frac{m}{1\cdot2\cdot3\cdot4\cdot5}+\frac{m+n}{3\cdot4\cdot5\cdot6\cdot7}+\frac{m+2n}{5\cdot6\cdot7\cdot8\cdot9}+\&c.\ldots=\frac{2m-3n}{6}\times S+\frac{25n-16m}{72}.$$

It

$$\text{If } m=1,\ n=0,\ \frac{1}{1.2.3.4.5}+\frac{1}{3.4.5.6.7}+\frac{1}{5.6.7.8.9}+\&c.\ldots=\frac{1}{3}S-\frac{2}{9};$$

$$m=1,\ n=1,\ \frac{1}{1.2.3.4.5}+\frac{2}{3.4.5.6.7}+\frac{3}{5.6.7.8.9}+\&c.\ldots=\frac{1}{8}-\frac{1}{6}S;$$

$$m=4,\ n=2,\ \frac{1}{1.2.3.5}+\frac{1}{3.4.5.7}+\frac{1}{5.6.7.9}+\&c.\ldots=\frac{1}{3}S-\frac{7}{36}.$$

$$m=25,\ n=16,\ \frac{25}{1.2.3.4.5}+\frac{41}{3.4.5.6.7}+\frac{57}{5.6.7.8.9}+\&c.\ldots=\frac{1}{3}S.$$

Let $r=2$, and we have $\frac{m}{1.3.5.7.9}+\frac{m+n}{5.7.9.11.13}+\frac{m+2n}{9.11.13.15.17}+\&c.\ldots=\frac{4m-5n}{192}\times S+\frac{59n-44m}{24.120}$.

$$\text{If } m=1,\ n=0,\ \frac{1}{1.3.5.7.9}+\frac{1}{5.7.9.11.13}+\frac{1}{9.11.13.15.17}+\&c.\ldots=\frac{1}{48}S-\frac{11}{24.30};$$

$$m=3,\ n=2,\ \frac{1}{1.5.7.9}+\frac{1}{5.9.11.13}+\frac{1}{9.13.15.17}+\&c.\ldots=\frac{1}{96}S-\frac{7}{24.60}.$$

Cor. If $n:m$ as $2r:2r+1$, the ſum of the ſeries can be accurately found; aſſume therefore $n=2r$, $m=2r+1$, and we have

$$\frac{1}{1.\overline{r+1}.\overline{3r+1}.\overline{4r+1}}+\frac{1}{\overline{2r+1}.\overline{3r+1}.\overline{5r+1}.\overline{6r+1}}+\&c.\ldots=\frac{1}{6.r.\overline{r+1}.\overline{2r+1}}.$$

$$\text{If } r=1,\ \frac{1}{1.2.4.5}+\frac{1}{3.4.6.7}+\frac{1}{5.6.8.9}+\&c.\ldots=\frac{1}{36};$$

$$r=3,\ \frac{1}{1.4.10.13}+\frac{1}{7.10.16.19}+\frac{1}{13.16.22.25}+\&c.\ldots\frac{1}{504}.$$

Having thus far explained the method of ſummation of ſuch ſeries as I propoſed to treat of in the firſt part of this paper, I truſt it is not neceſſary to ſay any thing further, as the ſame method of proceeding will manifeſtly continue the ſeries to any propoſed number of factors in the denominator; I ſhall therefore conclude with pointing out a remarkable property of thoſe ſeries whoſe ſum can be accurately found: that when the number of factors in the denominator is *even*, the numerator is always equal to the ſum of the two

middle factors; and when the number of factors be *odd*, the numerator will be equal to the middle factor, and consequently will take it out of the denominator, and leave a series whose numerators are unity, and whose denominators want the middle factor.

The method of summation of series here made use of may also be applied in investigating the sums of a great variety of other series; but as a further application of this method would carry us beyond the limits to which this paper must be confined, I shall re-assume the subject at some future opportunity, and proceed immediately to the second part.

PART II.

PROP.

To find the sum of the infinite series $\frac{p}{n \,.\, \overline{n+m} \ldots\ldots \overline{n+rm}} + \frac{q}{\overline{n+m} \ldots\ldots \overline{n+r+1 \,.\, m}} + \frac{s}{\overline{n+2m} \ldots\ldots \overline{n+r+2 \,.\, m}} +$ &c. *when the last differences of the numerators become equal to nothing.*

Assume $a + nb + n \,.\, \overline{n+m} \,.\, c + n \,.\, \overline{n+m} \,.\, \overline{n+2m} \,.\, d +$ &c. to any number (r) of terms; then, if for n we write $n+m$, $n+2m$, $n+3m$, &c. successively, there will result a series of quantities

 whose

whose $\acute{r}$th difference is $=0$; substitute, therefore, this series of quantities for p, q, s, &c. respectively, and the given series becomes

$$\frac{a+nb+n\cdot\overline{n+m}\cdot c+\&c.}{n\cdot\overline{n+m}\ldots\overline{n+rm}}+\frac{a+\overline{n+m}\cdot b+\overline{n+m}\cdot\overline{n+2m}\cdot c+\&c.}{\overline{n+m}\ldots\overline{n+r+1}\cdot m}+$$

$$\frac{a+\overline{n+2m}\cdot b+\overline{n+2m}\cdot\overline{n+3m}\cdot c+\&c.}{\overline{n+2m}\ldots\overline{n+r+2}\cdot m}+\&c.$$

which manifestly resolves itself into the following series

$$\frac{a}{n\cdot\overline{n+m}\ldots\overline{n+rm}}+\frac{a}{\overline{n+m}\ldots\overline{n+r+1}\cdot m}+\frac{a}{\overline{n+2m}\ldots\overline{n+r+2}\cdot m}+\&c.$$

$$\frac{b}{\overline{n+m}\ldots\overline{n+rm}}+\frac{b}{\overline{n+2m}\ldots\overline{n+r+1}\cdot m}+\frac{b}{\overline{n+3m}\ldots\overline{n+r+2}\cdot m}+\&c.$$

$$\frac{c}{\overline{n+2m}\ldots\overline{n+rm}}+\frac{c}{\overline{n+3m}\ldots\overline{n+r+1}\cdot m}+\frac{c}{\overline{n+4m}\ldots\overline{n+r+2}\cdot m}+\&c.$$

&c. &c. &c.

where the number of series is $\acute{r}$, the sum of each of which being taken by a well known rule, the sum of the given series becomes

$$\frac{a}{n\cdot\overline{n+m}\ldots\overline{n+r-1}\cdot m\cdot m\cdot r}+\frac{b}{\overline{n+m}\ldots\overline{n+r-1}\cdot m\cdot m\cdot\overline{r-1}}+$$

$$\frac{c}{\overline{n+2m}\ldots\overline{n+r-1}\cdot m\cdot m\cdot\overline{r-2}}+\&c.$$

where the law of continuation is manifest.

CASE I. *To find the sum of the infinite series* $\frac{3}{1\cdot 2\cdot 3\cdot 4}+\frac{6}{2\cdot 3\cdot 4\cdot 5}+\frac{10}{3\cdot 4\cdot 5\cdot 6}+\frac{15}{4\cdot 5\cdot 6\cdot 7}+\&c.$

Here $n=1$, $m=1$, $r=3$, and the third differences become $=0$; therefore $a+b+2c=3$, $a+2b+6c=6$, $a+3b+12c=10$, consequently $a=1$, $b=1$, $c=\frac{1}{2}$, and therefore the sum sought will be $\frac{1}{1\cdot 2\cdot 3\cdot 3}+\frac{1}{2\cdot 3\cdot 2}+\frac{1}{2\cdot 3}=\frac{11}{36}$.

CASE

CASE 2. *To find the ſum of the infinite ſeries* $\frac{1\cdot 2}{1\cdot 3\cdot 5\cdot 7}+\frac{2\cdot 3}{3\cdot 5\cdot 7\cdot 9}+\frac{3\cdot 4}{5\cdot 7\cdot 9\cdot 11}+$&c.

In this caſe $n=1$, $m=2$, $r=3$, and the 3d differences become $=0$; therefore $a+b+3c=2$, $a+3b+15c=6$, $a+5b+35c=12$, conſequently $a=\frac{3}{4}$, $b=\frac{1}{2}$, $c=\frac{1}{4}$, and hence the ſum of the required ſeries becomes $\frac{3}{1\cdot 3\cdot 5\cdot 2\cdot 3\cdot 4}+\frac{1}{3\cdot 5\cdot 2\cdot 2\cdot 2}+\frac{1}{5\cdot 2\cdot 4}=\frac{1}{24}$.

CASE 3. *To find the ſum of the infinite ſeries* $\frac{1}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}+\frac{2}{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}+\frac{4}{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}+\frac{12}{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}+\frac{31}{7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}+$&c.

Here $n=3$, $m=1$, $r=5$, and the 4th differences become $=0$; therefore $a+3b+12c+60d=1$, $a+4b+20c+120d=2$, $a+5b+30c+210d=4$, $a+6b+42c+336d=12$, conſequently $a=-54$, $b=47$, $c=-12$, $d=\frac{5}{6}$, therefore the ſum of the given ſeries is $\frac{-46}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 5}+\frac{47}{4\cdot 5\cdot 6\, 7\cdot 4}-\frac{12}{5\cdot 6\cdot 7\cdot 3}+\frac{5}{6\cdot 7\cdot 2\cdot 6}=\frac{61}{50400}$.

CASE 4. *To find the ſum of the infinite ſeries* $\frac{1}{1\cdot 4\cdot 7\cdot 10}+\frac{9}{4\cdot 7\cdot 10\cdot 13}+\frac{25}{7\cdot 10\cdot 13\cdot 16}+$&c.

Here $n=1$, $m=3$, $r=3$, and the 3d differences are $=0$; therefore $a+b+4c=1$, $a+4b+28c=9$, $a+7b+70c=25$, conſequently $a=\frac{1}{9}$, $b=\frac{-8}{9}$, $c=\frac{4}{9}$; $\therefore$ the ſum of the given ſeries will be $\frac{1}{1\cdot 4\cdot 7\cdot 3\cdot 3\cdot 9}-\frac{8}{4\cdot 7\cdot 3\cdot 2\cdot 9}+\frac{4}{7\cdot 3\cdot 9}=\frac{37}{2268}$.

CASE 5. *To find the Sum of the infinite Series* $\frac{1}{1\cdot3\cdot5\cdot7\cdot9}+\frac{4}{3\cdot5\cdot7\cdot9\cdot11}+\frac{10}{5\cdot7\cdot9\cdot11\cdot13}+\frac{20}{7\cdot9\cdot11\cdot13\cdot15}$.

In this case $n=1$, $m=2$, $r=4$, and the 4th differences become $=0$; therefore $a+b+3c+15d=1$, $a+3b+15c+105d=4$, $a+5b+35c+315d=10$, $a+7b+63c+693d=20$, consequently $a=\frac{5}{16}$, $b=\frac{3}{16}$, $c=\frac{1}{16}$, $d=\frac{1}{48}$, and hence the sum of the given series becomes $\frac{5}{1\cdot3\cdot5\cdot7\cdot2\cdot4\cdot16}+\frac{3}{3\cdot5\cdot7\cdot2\cdot3\cdot16}+\frac{1}{5\cdot7\cdot2\cdot2\cdot16}+\frac{1}{7\cdot2\cdot48}=\frac{1}{384}$.

This proposition may also be applied to find the sum of all those series whose numerators being unity, the denominators shall be deficient by any number of corresponding terms, however taken: for as the product of all such factors must form a progression, whose differences will become equal to nothing, if such products be assumed for the numerators of the given series having its factors compleated, another series will be formed equal to the given series, whose sum can be found by this proposition.

CASE 1. *To find the sum of the infinite series* $\frac{1}{1\cdot2\cdot4\cdot6}+\frac{1}{2\cdot3\cdot5\cdot7}+\frac{1}{3\cdot4\cdot6\cdot8}+$&c.

By completing the factors in the denominators, and multiplying the numerators by the same quantities the given series becomes $\frac{15}{1\cdot2\cdot3\cdot4\cdot5\cdot6}+\frac{24}{2\cdot3\cdot4\cdot5\cdot6\cdot7}+\frac{35}{3\cdot4\cdot5\cdot6\cdot7\cdot8}+$ &c. in which case $n=1$, $m=1$, $r=5$, and the 3d differences become $=0$;

$=0$; therefore $a+b+2c=15$, $a+2b+6c=24$, $a+3b+12c=35$, consequently $a=8$, $b=5$, $c=1$, and therefore the sum of the series required is $\frac{8}{1\cdot2\cdot3\cdot4\cdot5\cdot5}+\frac{5}{2\cdot3\cdot4\cdot5\cdot4}+\frac{1}{3\cdot4\cdot5\cdot3}=\frac{211}{7200}$.

CASE 2. *To find the sum of the infinite series* $\frac{1}{1\cdot5\cdot11}+\frac{1}{3\cdot7\cdot13}+\frac{1}{5\cdot9\cdot15}+$&c.

This series, when completed, becomes $\frac{189}{1\cdot3\cdot5\cdot7\cdot9\cdot11}+\frac{495}{3\cdot5\cdot7\cdot9\cdot11\cdot13}+\frac{1001}{5\cdot7\cdot9\cdot11\cdot13\cdot15}+\frac{1755}{7\cdot9\cdot11\cdot13\cdot15\cdot17}+$ &c. where $n=1$, $m=2$, $r=5$, and the 4th differences are $=0$; therefore $a+b+3c+15d=189$, $a+3b+15c+105d=495$, $a+5b+35c+315d=1001$, $a+7b+63c+693d=1755$, consequently $a=96$, $b=48$, $c=10$, $d=1$; and hence the sum of the given series is $\frac{96}{1\cdot3\cdot5\cdot7\cdot9\cdot2\cdot5}+\frac{48}{3\cdot5\cdot7\cdot9\cdot2\cdot4}+\frac{10}{5\cdot7\cdot9\cdot2\cdot3}+\frac{1}{7\cdot9\cdot2\cdot2}=\frac{487}{18900}$.

CASE 3. *To find the sum of the infinite series* $\frac{1}{1\cdot4\cdot13\cdot16}+\frac{1}{4\cdot7\cdot16\cdot19}+\frac{1}{7\cdot10\cdot19\cdot22}+$ &c.

This series resolves itself into $\frac{70}{1\cdot4\cdot7\cdot10\cdot13\cdot16}+\frac{130}{4\cdot7\cdot10\cdot13\cdot16\cdot19}+\frac{208}{7\cdot10\cdot13\cdot16\cdot19\cdot22}+$ &c. where $n=1$, $m=3$, $r=5$, and the 3d differences $=0$; therefore $a+b+4c=70$, $a+4b+28c=130$, $a+7b+70c=208$, from whence $a=54$, $b=12$, $c=1$; therefore the sum of the given series is

54

$$\frac{54}{1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 \cdot 3 \cdot 5} + \frac{12}{4 \cdot 7 \cdot 10 \cdot 13 \cdot 3 \cdot 4} + \frac{1}{7 \cdot 10 \cdot 13 \cdot 3 \cdot 3} = \frac{227}{163800}.$$

By this propofition we may alfo inveftigate the fum of the feries when there are any number of deficient terms in the denominators, and where the laft differences of the numerators become equal to nothing; for if the factors in the denominators be completed, and the numerators be multiplied by the fame quantities, their differences will ftill become equal to nothing.

CASE 1. *To find the fum of the infinite feries* $\frac{1}{1 \cdot 3 \cdot 4 \cdot 6} + \frac{3}{2 \cdot 4 \cdot 5 \cdot 7} + \frac{6}{3 \cdot 5 \cdot 6 \cdot 8} + \frac{10}{4 \cdot 6 \cdot 7 \cdot 9} + \frac{15}{5 \cdot 7 \cdot 8 \cdot 10} + \&c.$

This feries, by completing the factors in the denominators and multiplying the numerators by the fame quantities, becomes $\frac{10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{54}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{168}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c.$ in which cafe $n = 1$, $m = 1$, $r = 5$, and as the 5th differences are $= 0$; $\therefore a + b + 2c + 6d + 24e = 10$, $a + 2b + 6c + 24d + 120e = 54$, $a + 3b + 12c + 60d + 360e = 168$, $a + 4b + 20c + 120d + 840e = 400$, $a + 5b + 30c + 210d + 1680e = 810$, from whence $a = 0$, $b = 0$, $c = -1$, $d = 0$, $e = \frac{1}{2}$, confequently the fum of the given feries is $= -\frac{1}{3 \cdot 4 \cdot 5 \cdot 3} + \frac{1}{5 \cdot 2} = \frac{17}{180}$.

CASE 2. *To find the fum of the infinite feries* $\frac{1}{1 \cdot 3 \cdot 7 \cdot 9} + \frac{5}{3 \cdot 5 \cdot 9 \cdot 11} + \frac{11}{5 \cdot 7 \cdot 11 \cdot 13} + \frac{19}{7 \cdot 9 \cdot 13 \cdot 15} + \&c.$

By proceeding as before this feries becomes $\frac{5}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9} +$

35

$\frac{35}{3\cdot5\cdot7\cdot9\cdot11}+\frac{99}{5\cdot7\cdot9\cdot11\cdot13}+$&c. where $n=1$, $m=2$, $r=4$, and the 4th differences $=0$; therefore $a+b+3c+15d=5$, $a+3b+15c+105d=35$, $a+5d+35c+315d=99$, $a+7b+63c+693d=209$, consequently $a=-1$, $b=\frac{3}{4}$, $c=\frac{1}{2}$, $d=\frac{1}{4}$, therefore the sum of the given series $=\frac{-1}{1\cdot3\cdot5\cdot7\cdot2\cdot4}+\frac{3}{3\cdot5\cdot7,2\cdot3\cdot4}+\frac{1}{5\cdot7\cdot2\cdot2\cdot2}+\frac{1}{7\cdot2\cdot4}=\frac{3}{140}$.

By a method similar to that made use of in this proposition may any number of factors be taken from the denominators of those series delivered in part the first, and also from a great variety of others; but as the examples here given must be sufficient to point out the method of proceeding in all other cases, we will proceed to the third part.

PART III.

THE sum of every converging infinite series, whose terms ultimately become equal to nothing, may always be exhibited by the sum of another series formed by collecting two or more terms of the former series into one. This is not true, however, where the terms of the infinite series continally diverge, or converge to any assignable quantity, and

and are affected with the signs +, −, alternately: for instance, the series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} +$ &c. if we collect two terms into one, beginning at the first term, will become $-\frac{1}{2\cdot 3} - \frac{1}{4\cdot 5} - \frac{1}{6\cdot 7} +$ &c. If we begin at the second term it becomes $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} +$ &c.; neither of which gives the sum of the assumed series; but in this, and every other case of the like nature, a correction will be necessary: to determine the value of which, and from whence the necessity thereof arises, is the subject of this third part.

L E M M A.

If r be any quantity whatever, then will $\frac{1}{2r} = \frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} +$ &c. *ad infinitum.*

For $\frac{1}{2r} = \frac{1}{r+r} =$ (by common division) $\frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} +$ &c. ad infinitum.

Cor. 1. Hence $-\frac{1}{2r} = -\frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \frac{1}{r} -$ &c. ad infinitum.

Cor. 2. Hence also $\frac{z}{2v} = \frac{z}{v} - \frac{z}{v} + \frac{z}{v} - \frac{z}{v} +$ &c. ad infinitum;

and $-\frac{z}{2v} = -\frac{z}{v} + \frac{z}{v} - \frac{z}{v} + \frac{z}{v} -$ &c. ad infinitum.

P R O P. I.

If $\frac{rn+m}{n}$ *be the general term of a series formed by writing for n any series of numbers in arithmetic progression, and whose signs are alternately + and −; then if a series be formed by col-*

 lecting

lecting two terms into one, beginning at the first term, the sum of the series thence arising will be less than the sum of the given series by $\frac{1}{2r}$. *If a series be formed by beginning at the second Term, the sum thereof will be greater than the sum of the given series by* $\frac{1}{2r}$.

For let $\frac{n}{rn+m} - \frac{n+a}{\overline{n+a}\,.\,r+m}$ be any two successive terms of the series, which, if we begin to collect at the first term (the first term being $+$) will be two terms to be collected into one, and which will therefore give $\frac{-am}{\overline{rn+m} \times \overline{n+a}\,.\,r+m}$ for a general term of the resulting series. Let us now make n infinite, and then the denominator of this term becomes infinite, and the numerator finite; therefore the terms of this latter series at an infinite distance becoming infinitely small, the series will there terminate. Now, by making n infinite in the given series, the two successive general terms at an infinite distance become $\frac{1}{r} - \frac{1}{r}$; consequently *this* series is still continued after the *other* terminates; and the terms of such a continuation will be (as they begin with $\frac{n}{rn+m} - \frac{n+a}{\overline{n+a}\,.\,r+m}$ by making n infinite) $\frac{1}{r} - \frac{1}{r} + \frac{1}{r} - \frac{1}{r} + \&c.$ which will also be continued ad infin. and whose sum by the lemma is $\frac{1}{2r}$; consequently the given series exceeds that which is formed by collecting two terms into one, beginning at the first, by $\frac{1}{2r}$; hence the sum of the *latter* series $+\frac{1}{2r}$ will be equal to the sum of the *former*. If we begin to

collect at the second term, then will $-\frac{n}{rn+m}+\frac{n+a}{\overline{n+a}.r+m}$ be the two successive general terms of the given series to be collected into one; consequently the continuation of the given series when n becomes infinite will be $-\frac{1}{r}+\frac{1}{r}-\frac{1}{r}+\frac{1}{r}-$ &c. ad infinitum whose sum, by cor. 1. to the lem. is $-\frac{1}{2r}$; in this case, therefore, the sum of the given series is less than the sum of the series formed by collecting two terms into one, beginning at the second term, by $\frac{1}{2r}$; hence the sum of the *latter* series $-\frac{1}{2r}$ will be equal to the sum of the *former*.

CASE 1. *Let the given series be* $\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+$&c.

Here $r=1$, $n=1, 2, 3, 4,$ &c. and $m=1$. Now, if we begin to collect at the first term, the series resolves itself into $-\frac{1}{2.3}-\frac{1}{4.5}-\frac{1}{6.7}-$ &c. and the correction, to be *added*, being $\frac{1}{2}$, we have $-\frac{1}{2.3}-\frac{1}{4.5}-\frac{1}{6.7}-$&c. $+\frac{1}{2}$ for the sum of the given series. Now $-\frac{1}{2.3}-\frac{1}{4.5}-\frac{1}{6.7}-$ &c. is well known to be equal to $-1+$ hyp. log. of 2; consequently the sum of the given series is $=-\frac{1}{2}+$ hyp. log. of 2.

If we begin to collect at the second term, the series becomes $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+$ &c. and the correction, to be *subtracted*, being $\frac{1}{2}$, we have $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+$ &c. $-\frac{1}{2}$ for the sum of the given series;

series; but $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+$ &c. is equal to the hyp. log. of 2; therefore the sum of the given series is $= -\frac{1}{2}$ + hyp. log. of 2, the same as before.

CASE 2. *Let the given series be* $\frac{1}{3}-\frac{2}{5}+\frac{3}{7}+\frac{4}{9}+$ &c.

Here $r=2$, $n=1, 2, 3, 4,$ &c. $m=1$. Now, if we begin to collect at the second term, the series becomes $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$ &c. and the correction, to be *subtracted*, being $\frac{1}{4}$, we have $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$ &c. $-\frac{1}{4}$ for the sum of the given series; but $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+$ &c. is equal to a circular arc (A) of $22°\frac{1}{2}$, whose radius is unity; therefore the sum of the given series $= A - \frac{1}{4}$.

P R O P. II.

If $\frac{x+nz}{w+nv}$ *be the general term of a series formed by writing for* n *any series of numbers in arithmetic progression, and whose terms are alternately* + *and* −; *then if a series be formed by collecting two terms into one, beginning at the first term, the sum of the series thence arising will be less than the sum of the given series by* $\frac{z}{2v}$. *If a series be formed by beginning at the second term, the sum thereof will be greater than the sum of the given series by* $\frac{z}{2v}$.

For let $\frac{x+nz}{w+nv} - \frac{x+\overline{n+a}\,.\,z}{w+\overline{n+a}\,.\,v}$ be any two ſucceſſive terms of the given ſeries, which, if we begin to collect at the firſt term, will be the two general terms to be collected into one, and will therefore give $\frac{axv-awz}{\overline{w+nv}\times\overline{w+n+a\,.\,v}}$ for a general term of the reſulting ſeries. Let us now make n infinite and then this term will vaniſh, and conſequently the reſulting ſeries will terminate at an infinite diſtance. Now, by making n infinite in the given ſeries, the two ſucceſſive terms (as they begin with $\frac{x+nz}{w+nv} - \frac{x+\overline{n+a}\,.\,z}{w+\overline{n+a}\,.\,v}$ by making n infinite) become $\frac{z}{v} - \frac{z}{v}$; this ſeries, therefore, is ſtill continued after the other terminates; and the terms of ſuch a continuation will be $\frac{z}{v} - \frac{z}{v} + \frac{z}{v} - \frac{z}{v}$ + &c. ad infinitum, and whoſe ſum by cor. 2. to the lem. is $\frac{z}{2v}$; conſequently the given ſeries exceeds that which is formed by collecting two terms into one, beginning at the firſt, by $\frac{z}{2v}$; hence the ſum of the *latter* ſeries $+\frac{z}{2v}$ will be equal to the ſum of the *former*. Now, if we begin to collect at the ſecond term, then will $-\frac{x+nz}{w+nv} + \frac{x+\overline{n+a}\,.\,z}{w+\overline{n+a}\,.\,v}$ be two general terms of the given ſeries to be collected into one; conſequently the continuation of the given ſeries, when n becomes infinite, will be $-\frac{z}{v} + \frac{z}{v} - \frac{z}{v} + \frac{z}{v}$ – &c. ad infinitum, whoſe ſum by cor. 2. to the lem. is $-\frac{z}{2v}$; in this caſe, therefore, the ſum of the given ſeries is leſs than the ſum of the ſeries formed by collecting two terms into one, beginning at the ſecond term, by

z

$\frac{z}{2v}$; hence the ſum of the *latter* ſeries $-\frac{z}{2v}$ will be equal to the ſum of the *former*.

CASE 1. *Let the given ſeries be* $\frac{7}{3}-\frac{11}{5}+\frac{15}{7}-\frac{19}{9}+$&c.

Here $x=3$, $z=2$, $w=1$, $v=1$, $n=2, 4, 6, 8$, &c. Now, if we begin to collect at the firſt term, the ſeries becomes $\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\frac{2}{11\cdot 13}+$&c. and the correction, to be *added*, being 1, we have $\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\frac{2}{11\cdot 13}+$&c. $+1$ for the ſum of the given ſeries; but if $A=$ a circular arc of $45°$ whoſe radius is unity, it is well known that $\frac{2}{3\cdot 5}+\frac{2}{7\cdot 9}+\frac{2}{11\cdot 13}+$ &c. $=1-A$; therefore the ſum of the given ſeries is $2-A$.

CASE 2. *Let the given ſeries be* $\frac{16}{1}-\frac{27}{2}+\frac{38}{3}-\frac{49}{4}+$&c.

Here $w=1$, $v=1$, $x=16$, $z=11$, $n=0, 1, 2, 3$, &c. Now, if we begin to collect at the firſt term, the ſeries becomes $\frac{5}{1\cdot 2}+\frac{5}{3\cdot 4}+\frac{5}{5\cdot 6}+$&c. and the correction, to be *added*, being $\frac{11}{2}$, we have $\frac{5}{1\cdot 2}+\frac{5}{3\cdot 4}+\frac{5}{5\cdot 6}+$&c. $+\frac{11}{2}$ for the ſum of the given ſeries; but $\frac{5}{1\cdot 2}+\frac{5}{3\cdot 4}+\frac{5}{6\cdot 7}+$&c. is equal to $5\times$ hyp. log. of 2, conſequently the ſum of the given ſeries is equal to $\frac{11}{2}+5\times$ hyp. log. of 2.

Becauſe $\frac{axv-awx}{\overline{w+nv}\times\overline{w+n+a\cdot v}}$, the general term of the ſeries

formed

formed by reducing two terms into one, has its numerator independent of the value of, n, it is manifeſt, that the numerators of that ſeries will be all equal. Now, if a ſeries be aſſumed, the numerators of whoſe terms are unity, and in every other reſpect the ſame as the ſeries in this propoſition, that is, if $\frac{1}{w+nv} - \frac{1}{w+\overline{n+a}\,.\,v}$ be two ſucceſſive terms of a ſeries, it is manifeſt, that if every two terms of this ſeries be reduced into one, the general term of the reſulting ſeries will be $\frac{-av}{\overline{w+nv}\times\overline{w+\overline{n+a}\,.\,v}}$, where the numerator is a conſtant quantity $-av$; conſequently the ſum of the ſeries whoſe general term is $\frac{avx-awz}{\overline{w+nv}\times\overline{w+\overline{n+a}\,.\,v}}$ is to the ſum of the ſeries whoſe general term is $\frac{-av}{\overline{w+nv}\times\overline{w+\overline{n+a}\,.\,v}}$ as $vx-wz$ to $-v$, or in a given ratio; whenever, therefore, the ſum of the latter ſeries can be found, the ſum of the former can be found, and conſequently, after proper correction, the ſum of the ſeries in this propoſition can be found.

Hence, therefore, in the two caſes given above in whatever arithmetic progreſſion the numerators may proceed, the ſum of the former can always be expreſſed by circular arcs, and the latter by the hyp. log. of 2.

Hence alſo, as it appears from lem. 1. part the firſt, that the ſum of the ſeries $\frac{1}{1} - \frac{1}{r+1} + \frac{1}{2r+1} - \frac{1}{3r+1} +$ &c. can always be expreſſed by circular arcs and logarithms, it is manifeſt, that if the numerators form any arithmetic progreſſion, the ſum of ſuch ſeries may be found by this propoſition, and will always be exhibited by circular arcs and logarithms.

Beſides

Besides the series contained in the foregoing propositions, a great variety of other series might be produced where a correction is necessary, after collecting two terms into one, in order to exhibit the true value of the given series. As the proper correction, however, may always be found from the principles delivered in the above propositions, that is, by considering what the terms of the given series become at an infinite distance, I shall only add one or two instances more, and conclude what I at present intend to offer on this subject.

EX. 1. *Let it be required to find the sum of the infinite series* $\frac{3 \cdot 4}{1 \cdot 2} - \frac{4 \cdot 5}{2 \cdot 3} + \frac{5 \cdot 6}{3 \cdot 4} - \frac{6 \cdot 7}{4 \cdot 5} + \&c.$

This, by resolving two terms into one, becomes $\frac{16}{1 \cdot 2 \cdot 3} + \frac{24}{3 \cdot 4 \cdot 5} + \frac{32}{5 \cdot 6 \cdot 7} - \&c.$; and as the terms of the given series continually approach to unity, the correction, to be *added*, is $\frac{1}{2}$, consequently $\frac{16}{1 \cdot 2 \cdot 3} + \frac{24}{3 \cdot 4 \cdot 5} + \frac{32}{5 \cdot 6 \cdot 7} - \&c. + \frac{1}{2}$ is equal to the sum of the given series; but by prop. 1. part I. the sum of the series $\frac{16}{1 \cdot 2 \cdot 3} + \frac{24}{3 \cdot 4 \cdot 5} + \frac{32}{5 \cdot 6 \cdot 7} + \&c.$ is equal to $8S - 2$ (S being the hyp. log. 2.) consequently the sum of the given series is $8S - 1\frac{1}{2}$.

EX. 2. *Let it be required to find the sum of the infinite series* $\frac{1 \cdot 2}{1 \cdot 3} - \frac{2 \cdot 3}{3 \cdot 5} + \frac{3 \cdot 4}{5 \cdot 7} - \frac{4 \cdot 5}{7 \cdot 9} + \&c.$

This series, by resolving two terms into one, becomes $\frac{4}{1 \cdot 3 \cdot 5} + \frac{8}{5 \cdot 7 \cdot 9} + \frac{12}{9 \cdot 11 \cdot 13} + \&c.$ and as the terms of the given

ſeries continually approach to $\frac{1}{4}$, the correction, to be *added*, will be $\frac{1}{8}$, therefore $\frac{4}{1 \cdot 3 \cdot 5} + \frac{8}{5 \cdot 7 \cdot 9} + \frac{12}{9 \cdot 11 \cdot 13} + \text{\&c.} + \frac{1}{8}$ is = to the ſum of the given ſeries; but by prop. 1. part I. the ſum of $\frac{4}{1 \cdot 3 \cdot 5} + \frac{8}{5 \cdot 7 \cdot 9} + \frac{12}{9 \cdot 11 \cdot 13} + \text{\&c.}$ is equal to $\frac{1}{4}S + \frac{1}{8}$ (S being a circular arc of 45°, whoſe radius is unity) hence the ſum of the given ſeries is $\frac{1}{4}S + \frac{1}{4}$.

This method is not only applicable to thoſe caſes, where the given ſeries reſolves itſelf into another, whoſe ſum is either accurately known or can be expreſſed by circular arcs and logarithms, but alſo to thoſe caſes where we want to approximate to the value of the given ſeries, as it muſt, in general, be neceſſary firſt to render the terms of the ſeries converging, by collecting two into one, before the operation of approximation begins, and conſequently a correction of this latter is neceſſary in order to exhibit the value of the given ſeries.

E R R A T A.

Page. Line.

72. 2. *for* comunicated *read* communicated

77. 12. *for* mercury removed without cauſing this to riſe *read* remove without cauſing the mercury to riſe

81. 9. *for* ſmaller towards the middle *read* larger towards the middle

391. — the laſt line, *for* $-$ *read* $+$

403. 12. *for* -54 *read* -46

408. 3. from the bottom, *for* $\frac{rn+m}{n}$ *read* $\frac{n}{rn+m}$.